§2.3 Counting Techniques

Recall: If Sample Space S consists of equiprobable outcomes, then

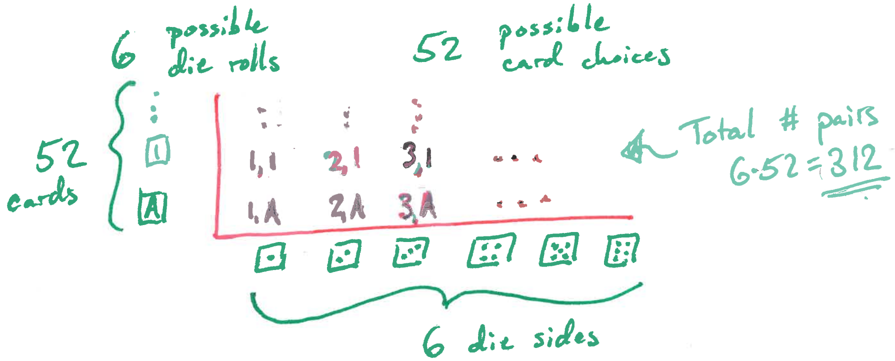
$$P(E) = \frac{\#\text{ outcomes in } E}{\#\text{ outcomes in } S}$$

Because of this many basic probability computations are completed by counting.

Method 1 Ordered Pairs (Tables)

Example: Experiment is to roll a die & pick a card

How many possible events are there?



Fact: $\begin{pmatrix}\#\text{ outcomes in product}\\ \text{sample space } S_1 \times S_2\end{pmatrix} = \begin{pmatrix}\#\text{ outcomes}\\ \text{in } S_1\end{pmatrix} \cdot \begin{pmatrix}\#\text{ outcomes}\\ \text{in } S_2\end{pmatrix}$

Method 2 Decision Trees

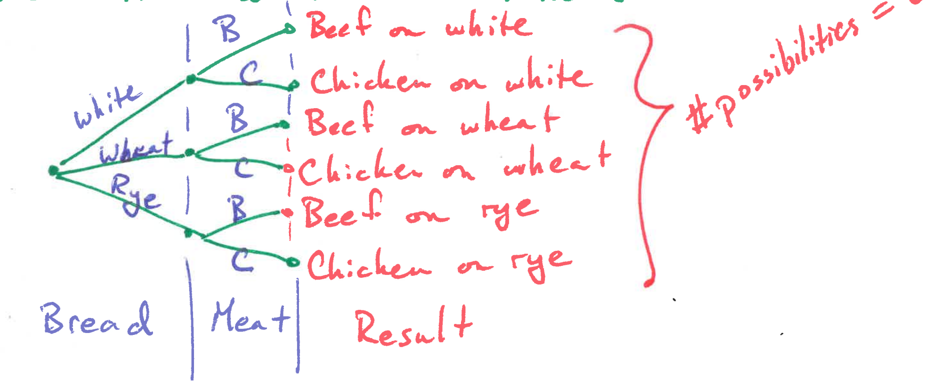
Sometimes it is useful to organize this as a decision tree.

Example: Store has sandwiches with

3 types of bread (White, Wheat, Rye)

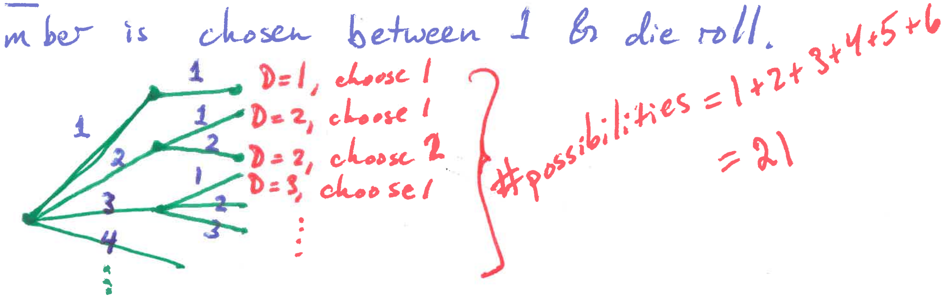
2 types of meat (beef, chicken)

In this case there are $3 \cdot 2 = 6$ types of sandwich possible. We can organize them into a "tree" as follows



Decision trees can be helpful to count things which aren't direct products

Example: A die is rolled and then a number is chosen between 1 & die roll.



Method 3 Permutations & Combinations

Suppose each outcome is a choice of k things picked from n total possibilities.

In this case the count depends on whether choice is

- exclusive ("without replacement") → Every item chosen must be different
- ordered → Different orderings count as different choices.

Example: Ali, Burak, and Cem each choose a kebab from menu of Beef, Chicken, Mixed, and Doner. How many choices are possible?

Formula: n^k

inclusive ("with replacement") because all three could choose same type

ordered because Ali doesn't want to eat Cem's food.

$$\boxed{\begin{aligned}\#\text{Choices} &= (4 \text{ for Ali})\cdot(4 \text{ for Burak})\cdot(4 \text{ for Cem})\\ &= \underbrace{4\cdot 4\cdot 4}_{3\ \#\text{s}} = 4^3 = 64\end{aligned}}$$

Example: 1st, 2nd, and 3rd ranked teams will be chosen from group of 6 teams. #Choices = ?

Formula: $\underbrace{n\cdot(n-1)\cdots(n-k+1)}_{k\ \#\text{s}}$

exclusive ("without replacement") because 1st place team cannot also be 3rd place

ordered because rank matters.

$$\boxed{\begin{aligned}\#\text{Choices} &= (6 \text{ for } 1^{st})\cdot(5 \text{ for } 2^{nd})\cdot(4 \text{ for } 3^{rd})\\ &= \underbrace{6\cdot 5\cdot 4}_{3\ \#\text{s}} = 120\end{aligned}}$$

↖ sometimes this is written $\frac{6!}{3!}$

Example: Pick a "hand" of 5 cards from deck of 52. #Choices = ?

Formula: $\binom{n}{k}$

exclusive ("without replacement") because no card can be picked twice

unordered because order of picking doesn't matter

$$\boxed{\begin{aligned}\#\text{Choices} &= \frac{(52 \text{ for } 1^{st})\cdot(51 \text{ for } 2^{nd})\cdots(48 \text{ for } 5^{th})}{\#\text{Orders of 5 Cards}}\\ &= \frac{(52)(51)(50)(49)(48)}{\underbrace{(1)(2)(3)(4)(5)}_{5\ \#\text{s}}} = 2{,}598{,}960\end{aligned}}$$

↑ sometimes this is written $\frac{52!}{5!\,47!} = \binom{52}{5}$

Example 5 kebabs are chosen from menu of Beef, Chicken, Mixed, Doner. #Choices = ?

Formula: $\binom{n+k-1}{k}$

inclusive because all 5 could be same

unordered because order of choice does not matter.

$$\boxed{\#\text{Choices} = \frac{(8)(7)(6)(5)(4)}{\underbrace{(1)(2)(3)(4)(5)}_{5\ \#\text{s}}} = 56}$$

← Partition 5 items into 4 parts (allowing empty parts)